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Mathematical modeling of water dynamics in soils – a tool for smart management of irrigation networks.

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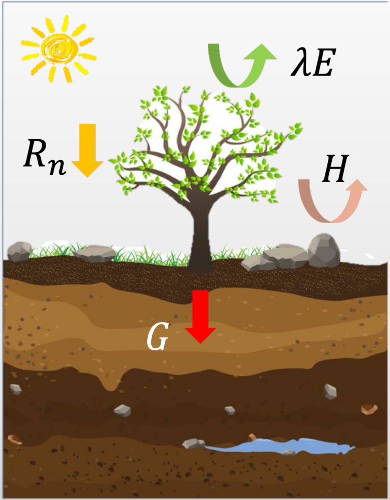
Motivation

- **knowledge of** moisture + nutrient transport dynamics in soils is governed by
  - plant transpiration
  - water evaporation
  - climatic conditions
- is crucial for estimation of water demand of crop from **irrigation** for
  - **short time (changing weather conditions)**
  - **long time (changing climatic conditions)**



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What can be understood by "Modeling of dynamic processes"?

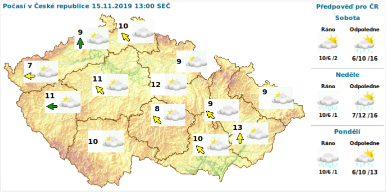
e.g. weather models ≡

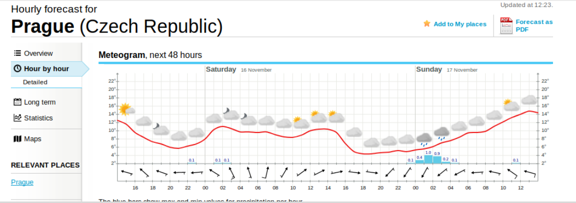
(the results we study every day)

- temperature $(\vec{x}, t)$
- wind speed $(\vec{x}, t)$
- precipitation $(\vec{x}, t)$
- cloudiness $(\vec{x}, t)$

where

- $\vec{x}$ denotes location
- t denotes time





source: http://yr.no, http://chmi.cz

Modeling of dynamic processes

climatic models ≡

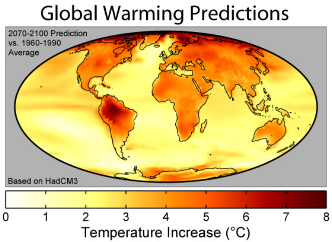
≡ **average values for certain period for certain larger location**

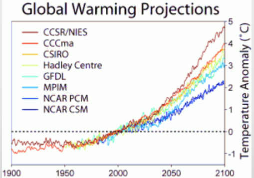
- temperature $(\vec{x}, t)$,
- precipitation $(\vec{x}, t)$

where $t \in \{\text{several decades}\}$,
and $\vec{x} \in \{\text{regional} \rightsquigarrow \text{continental scale}\}$

$$\Downarrow$$

- increased temperature $\rightsquigarrow$ more heat energy in atmosphere $\rightsquigarrow$ increased dynamics of all weather processes



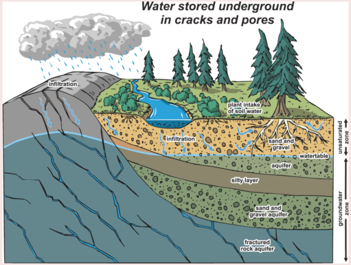


source: https://blogs.ei.columbia.edu

Soil hydrodynamical models

Quantitative modeling

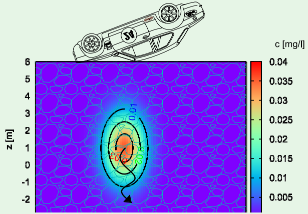
- groundwater table $(\vec{x}, t)$
- soil moisture (water content) $(\vec{x}, t)$
- overland flow (surface runoff) $(\vec{x}, t)$
- soil temperature $(\vec{x}, t)$



source: Brubacher, Jordan. (2015).

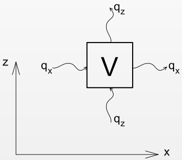
Qualitative modeling

- solute concentration $(\vec{x}, t)$ (typically heavy metals, radionuclids, fertilizers, etc.)
- concentration of NAPLs $(\vec{x}, t)$ (typically oil products)



How are these models solved

- Mathematical model is indeed just some mathematical equation.
- Dynamical processes are since Newton expressed by differential equations.
- Models typically originates from application of law of mass conservation.



$$\frac{\partial V}{\partial t} = -\nabla \cdot \vec{q}$$

this simply equation + some tricks

- mathematical model of weather/climate
- mathematical model of soil hydrodynamics (quantity + quality models)
- + way more

Modeling water dynamics in soils with evaporation

- after some tricks with equation of mass conservation + extra physical assumptions $\rightsquigarrow$ set of **thermodynamic (TDE)** + **hydrodynamic (HDE)** equations

HDE:

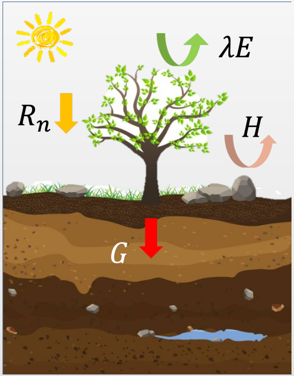
Δ wat. content | water flow due pressure + temp grad. | Δ vap. content

$$\frac{\partial \theta}{\partial t} = \nabla \cdot \mathbb{K}_{Th} \nabla h + \nabla \cdot K_{TT} \nabla T - \frac{\partial \theta_v}{\partial t}$$

TDE:

Δ heat energy | heat flux | latent heat source

$$C_T \frac{\partial T}{\partial t} = \nabla \cdot \mathbb{K}_{Th} \nabla h + \nabla \cdot \lambda_{TT} \nabla T + \nabla \cdot C_l \mathbb{K}_{lh} T \nabla z - L_f \varrho \frac{\partial \theta_v}{\partial t}$$



source: R. G. Allen, et al. (1998)

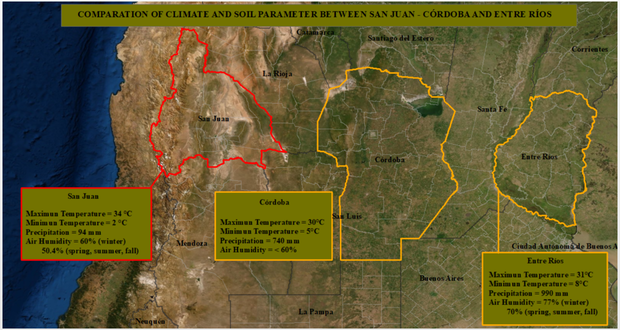
Importance of water

- water has "climate/weather stabilization effect", because
  - huge specific heat capacity $C = \frac{\mathrm{d}Q}{\mathrm{d}T} = 4188\ \mathrm{J.kg^{-1}.K^{-1}}$
  - huge latent heat $L_f = 3.337 \times 10^5\ \mathrm{J.kg^{-1}}$

e.g. latent heat source in TDE $\left[-\varrho L_f \frac{\partial \theta_v}{\partial t}\right]$ [W]

EXAMPLE: $1\mathrm{m}^3 \rightsquigarrow 1.0001\mathrm{m}^3$ due condensation within 1 min, and so $\Delta\theta_v = -1 \times 10^{-4}\ \mathrm{m}^3$

$$-\frac{-1 \times 10^{-4}}{60} \times 3.337 \times 10^5 \times 1000 \approx 600\ \mathrm{W!!}$$



condensation of 0.1 l in a volume of $1\mathrm{m}^3$ within 1 min. generates a powersource of 600 W

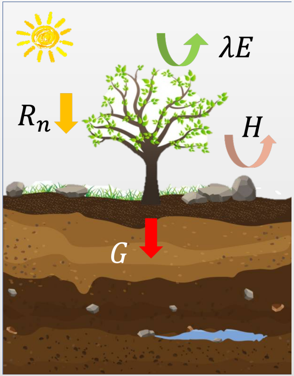
Model summary

with this model we are able to combine

- evaporation on the surface
- evaporation inside in the soil profile
- crop water consumption
- different irrigation schemes

with the input representing

- soil type conditions
- climatic conditions



and the model is capable to provide space $\times$ time distribution of soil humidity

How is this model solved

- the model is expressed by some differential equation
- particularly by this

$$\frac{\partial V}{\partial t} = -\boldsymbol{\nabla} \cdot \vec{q}$$

$$\downarrow$$

$$C_T \frac{\partial T}{\partial t} = \nabla \cdot \mathbb{K}_{Th} \nabla h + \nabla \cdot \lambda_{TT} \nabla T + \nabla \cdot C_l \mathbb{K}_{lh} T \nabla z - L_f \varrho \frac{\partial \theta_v}{\partial t}$$

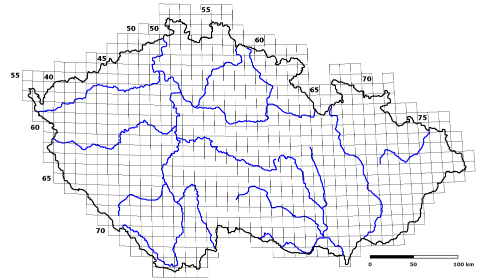
- maybe we remember that differential equation are solved by direct integration
- **which is impossible for such models as weather or soil hydrodynamics**

How to proceed with solution?

Grid model representation

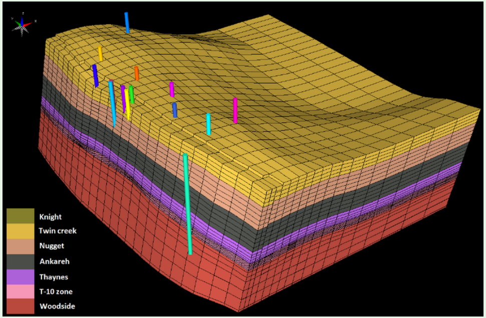
The most simplistic way of computer solution of differential equations is a "**grid representation**" (finite difference method)

Grid for weather forecast



source `http://www.chmi.cz`

Grid for catchment soil water dynamics



source: Chen, et al. (2013)

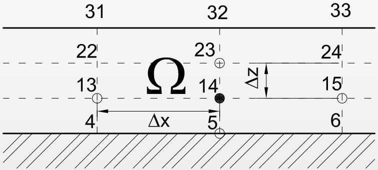
Grid model representation – finite difference approximation

1^{st} derivative :

$$\frac{\partial h}{\partial x} \approx \frac{h_{15} - h_{14}}{\Delta x}$$
$$\frac{\partial h}{\partial z} \approx \frac{h_{23} - h_{14}}{\Delta z}$$

2^{nd} derivative :

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_{15} - 2h_{14} + h_{13}}{(\Delta x)^2}$$
$$\frac{\partial^2 h}{\partial z^2} \approx \frac{h_{23} - 2h_{14} + h_{5}}{(\Delta z)^2}.$$



Short comment

Finite difference method = *archaism*. It is simle for explanation.
Modern **finite elements method** ⇝ similar discrete (grid) approach × ~~nodal values~~ ⇝ polynomial approximation coefficients.

Discrete model representation

- both weather and soil dynamic models are represented by

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \dots & + & a_{3n}x_n & = b_3 \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \dots & + & a_{nn}x_n & = b_n,
\end{array}
$$

??Can we model the ENTIRE planet like that?

- In theory YES
- In practise NO
- even with most recent supercomputers

IT4 Innovations – the biggest super computer in Central Europe

source: `http://it4innovations.cz`

Why we can't do that?

- Computer time needed to solve *system of linear equations* is analyzed as

$$T = c.\mathrm{NDOFs}^3$$

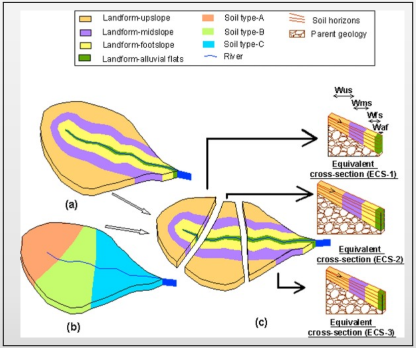
NDOFs	CPU time [s]
10^3	1
10^4	1000
10^5	1000000
10^6	1000000000
10^7	1000000000000
10^8	1000000000000000

Consequence

- Yes, we can create **SUPER accurate** weather forecast for the next 2 weeks
- But, the computation will take 2 years!!!! $\rightsquigarrow$ quite useless forecasting :(

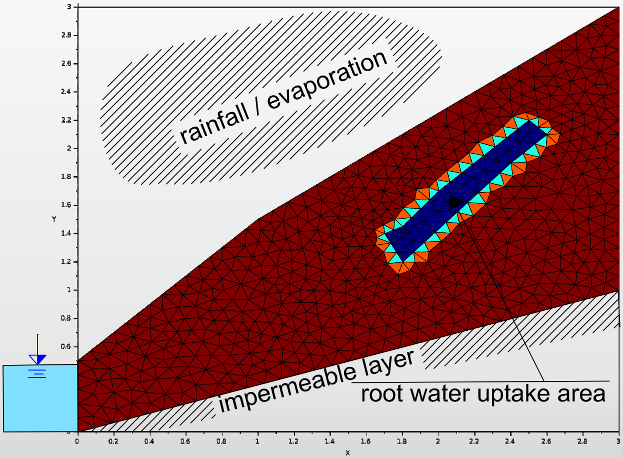
How to model efficiently

- we must search for simplifications
- e.g. representing a catchment by set of its representative crosssections

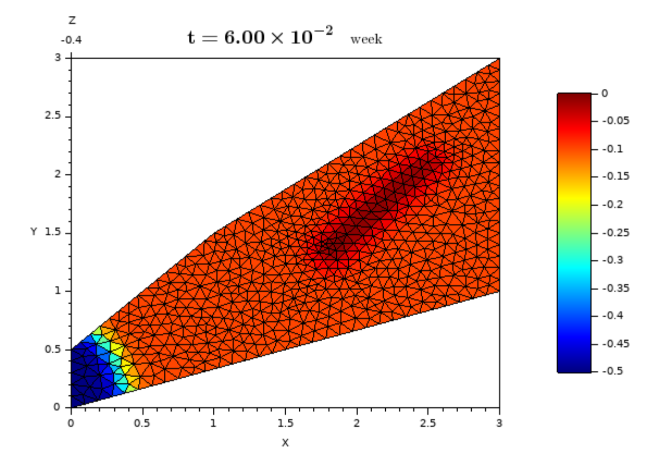


source: U. Khan, H. Ajami, N. K. Tuteja, A. Sharma, S. Kim: Catchment scale simulations of soil moisture dynamics using an equivalent cross-section based hydrological modelling approach, *Journal of Hydrology*(564), 2018

Another example of 2D representative cross-section model



Example of 2D representative cross-section model



Conclusions

Properties of the model

- with the proposed soil hydrodynamical model we are able to determine locations with moisture deficit
- such modeling of the entire catchment is possible in theory only, in practise we have to cope with simplifications

Application of the model

- weather forecast is a boundary condition which can be used for short time estimates of the soil moisture distribution in **near future**
- climate change models can be again used as an estimate of boundary conditions to estimate soil moisture for **further future** periods

Thank you for your attention.